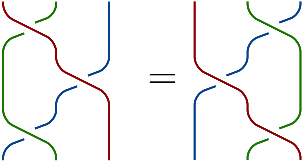
Yang–Baxter Equation I

Victoria LEBED, Université Caen Normandie



$(ab)c = a(bc)$

$z^{-1}(y^{-1}xy)z =$
$(z^{-1}y^{-1}z)(z^{-1}xz)(z^{-1}yz)$

T-Days III, Caen, October 2019

1 Avatars

Data: vector space V, $\sigma\colon V^{\otimes 2} \to V^{\otimes 2}$.

Yang–Baxter equation (YBE)

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2\colon V^{\otimes 3} \to V^{\otimes 3}$$

$\sigma_1 = \sigma \otimes \mathrm{Id}_V$, $\sigma_2 = \mathrm{Id}_V \otimes \sigma$

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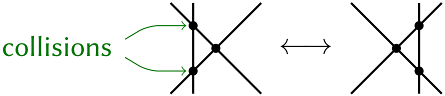
⇝ factorisation condition for the dispersion matrix in the 1-dim. n-body problem (*McGuire & Yang 60'*);



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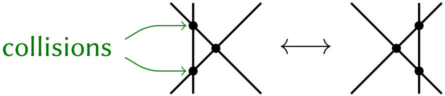

⇝ condition for the partition function in an exactly solvable lattice model (*Onsager '44; Baxter 70'*);

⇝ quantum inverse scattering method for completely integrable systems (*Faddeev et al. '79*);

⇝ factorisable S-matrices in 2-dim. QFT (*Zamolodchikov '79*);

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- R-matrices in quantum groups (*Drinfel′d 80'*);
- C^* algebras (*Woronowicz 80'*);
- twisted tensor product in non-commutative geometry (*Majid 90'*);
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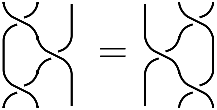
- twisted tensor product in non-commutative geometry (*Majid 90'*);
- rewriting systems;
- braid equation in low-dimensional topology.

$\sigma \longleftrightarrow$ 

YBE $\longleftrightarrow$ 

Reidemeister III move

2 Questions

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⤳ Find new solutions with properties interesting for different applications.

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- Classify solutions.

Hard! Solved only for $\dim V = 2$: 96 solutions, up to … (*Hietarinta '92*).

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- Understand the structure of the solution "variety".

2 Questions

Data: set V, $\sigma\colon V^{\times 2} \to V^{\times 2}$.

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$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2\colon V^{\times 3} \to V^{\times 3} \qquad \sigma_1 = \sigma \times \mathrm{Id}_V,\ \sigma_2 = \mathrm{Id}_V \times \sigma$$

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The two-step approach (*Drinfel′d 90'*):

Step 1. Classify set-theoretic solutions (called braided sets).

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Step 2. Study their deformations:

$$\text{braided sets} \quad \overset{\text{linearise}}{\rightsquigarrow} \quad \overset{\text{deform}}{\rightsquigarrow} \quad \text{linear solutions.}$$

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- Find solution invariants.

3 The flip and its deformation

Examples of braided sets:

✓ $\sigma(x, y) = (x, y)$;

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✓ $\sigma(x, y) = (y, x) \quad \rightsquigarrow \quad \sigma(x \otimes y) = y \otimes x + \hbar 1 \otimes [x, y]$,

where $(V, [\,])$ is a Lie algebra, and $\forall v, \; [1, v] = [v, 1] = 0$.

YBE for $\sigma \iff$ Jacobi identity for $[\,]$

4 Self-distributivity

✓ set S, binary operation $\triangleleft$, $\sigma(x,y) = (y, x \triangleleft y)$

YBE for σ $\iff$ self-distributivity for $\triangleleft$

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Examples:

⇝ group S with $x \triangleleft y = y^{-1}xy$;

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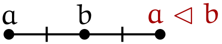
⇝ abelian group S, $t\colon S \to S$, $a \lhd b = ta + (1-t)b$.

5 SD: a historical digression

Self-distributivity: $(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$

① *Mituhisa Takasaki*, a fresh Japanese maths PhD in 1940 Harbin

Motivation: geometric symmetries.



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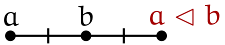

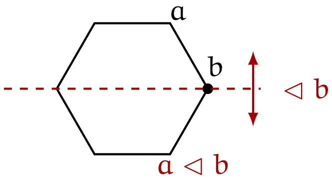

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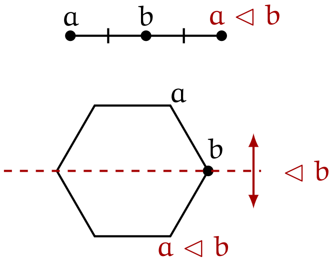

✓ More generally : abelian group A with $a \lhd b = 2b - a$.

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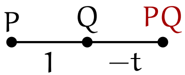

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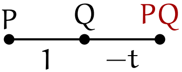

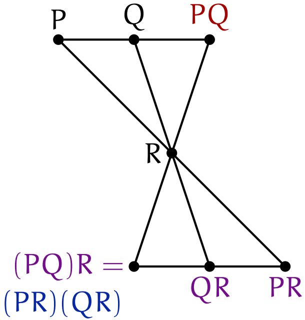

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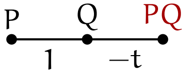

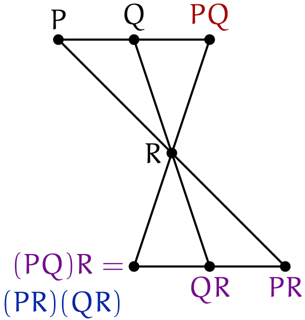

✓ Any group G with $g \lhd h = h^{-1}gh$, or $hg^{-1}h$, or …

5 SD: a historical digression

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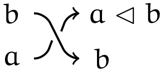
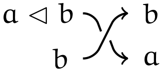
$(S, \lhd)$-colourings for braid diagrams:





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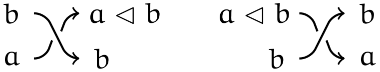

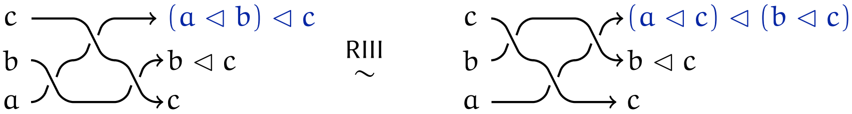

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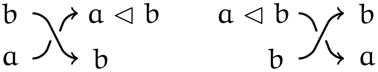
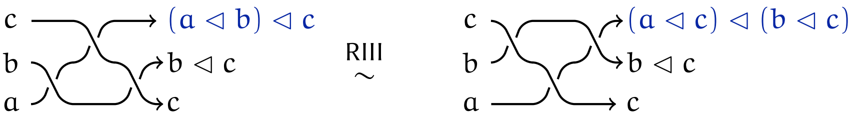
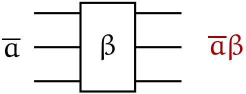




$\mathrm{End}(S^n) \leftarrow B_n^+$	RIII	$(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)$	shelf
$\mathrm{Aut}(S^n) \leftarrow B_n$	& RII	$\forall b,\ a \mapsto a \lhd b$ is bijective	rack
$S \hookrightarrow (S^n)^{B_n}$	& RI	$a \lhd a = a$	quandle

$a \mapsto (a, \dots, a)$

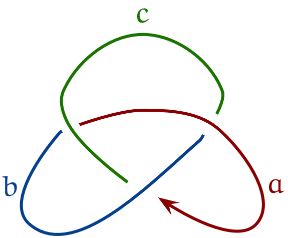


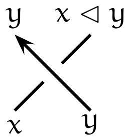
6 Braids and self-distributivity

S	$a \lhd b$	$(S, \lhd)$ is a	in braid theory
$_{\mathbb{Z}[t^{\pm 1}]}\mathrm{Mod}$	$ta + (1-t)b$	quandle	(red.) Burau: $B_n \to GL_n(\mathbb{Z}[t^{\pm}])$
group	$b^{-1}ab$	quandle	Artin: $B_n \hookrightarrow \mathrm{Aut}(F_n)$
twisted linear quandle			Lawrence–Krammer–Bigelow
$\mathbb{Z}$	$a + 1$	rack	$lg(w)$, $lk_{i,j}$
free shelf			Dehornoy: order on B_n

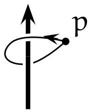
7 Knots and self-distributivity

$(S, \triangleleft)$-colourings for knot diagrams:





cf. Wirtinger presentation of $\pi_1(\mathbb{R}^3 \setminus K)$:



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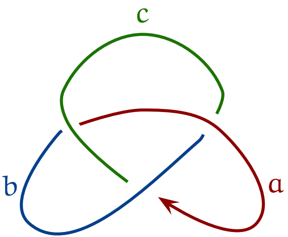

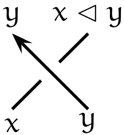

Proposition: $(S, \triangleleft)$ is a quandle $\Longrightarrow$

$\#\{\, (S, \triangleleft)$-colourings of diagrams $\}$ is a knot invariant.

7 Knots and self-distributivity

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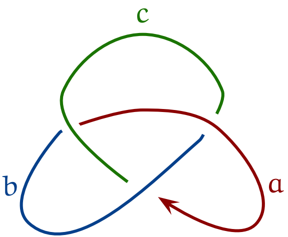

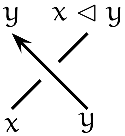

Proposition: $(S, \triangleleft)$ is a quandle $\implies$
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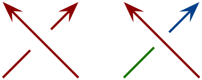
Example: $(\mathbb{Z}_3,\ a \triangleleft b = 2b - a)$



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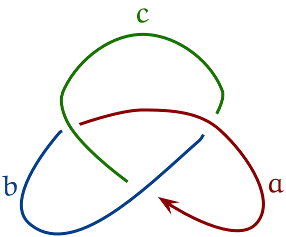

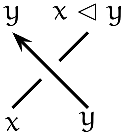

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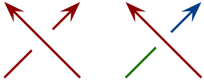
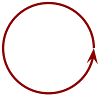
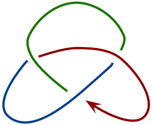




3 colourings



9 colourings

7 Knots and self-distributivity

Theorem (*Joyce & Matveev '82*):

$$\#\,\mathrm{Col}_{S,\triangleleft}(D) = \#\,\mathrm{Hom}_{\mathcal{Quandle}}(Q(K), S)$$

↝ $Q(K)$ = fundamental quandle of K

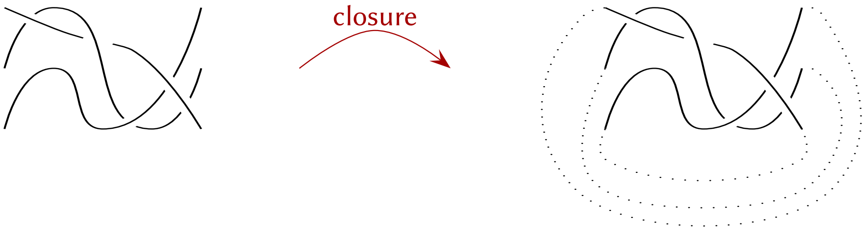
(a weak universal knot invariant);

7 Knots and self-distributivity

<u>Theorem</u> (*Joyce & Matveev '82*):

$$\#\,\mathrm{Col}_{S,\triangleleft}(D) = \#\,\mathrm{Hom}_{\mathcal{Quandle}}(Q(K), S) = \mathrm{Tr}(\rho_S(\beta))$$

- $Q(K)$ = fundamental quandle of K
  (a weak universal knot invariant);
- $\mathrm{closure}(\beta) = K$;
- $\rho_S : B_n \to \mathrm{Aut}(S^n)$ is the S-coloring invariant for braids.



8 Other applications of self-distributivity

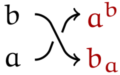
- study of large cardinals (*Laver & Dehornoy 90'*);
- Hopf algebra classification (*Andruskiewitsch–Graña '03*);
- integration of Leibniz (= generalised Lie) algebras (*Kinyon '07*);
- study of braided sets.

9 Upper strands matter!

Similarly, a braided set (+ extra axioms)

$\rightsquigarrow$ colouring invariants for braids and knots.

Diagram colorings by (S, σ):



Notation: $\sigma(a, b) = (b_a, a^b)$.

Example: $\sigma_{SD}(a, b) = (b, a \triangleleft b)$.

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All these braidings are idempotent: $\sigma\sigma = \sigma$.