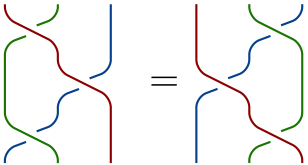
Yang–Baxter Equation II

Victoria LEBED, Université Caen Normandie



$(ab)c = a(bc)$

$z^{-1}(y^{-1}xy)z =$

$(z^{-1}y^{-1}z)(z^{-1}xz)(z^{-1}yz)$

T-Days III, Caen, October 2019

0 Previously...

Data: set S, $\sigma\colon S^{\times 2} \to S^{\times 2}$.

Yang–Baxter equation (YBE)

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \colon S^{\times 3} \to S^{\times 3} \qquad \sigma_1 = \sigma \times \mathrm{Id}_S,\ \sigma_2 = \mathrm{Id}_S \times \sigma$$

The two-step approach (*Drinfel′d 90'*):

Step 1. Classify set-theoretic solutions (called braided sets).

Step 2. Study their deformations:

$$\text{braided sets} \overset{\text{linearise}}{\rightsquigarrow} \overset{\text{deform}}{\rightsquigarrow} \text{linear solutions.}$$

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A cohomology theory for YBE solutions should:

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$$\sigma_\phi(x,y) = q^{\phi(x,y)}\sigma(x,y), \qquad \phi\colon S \times S \to \mathbb{Z} \text{ or } \mathbb{Z}_m.$$

ϕ a 2-cocycle $\implies$ σ_ϕ a linear YBE solution.

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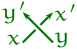
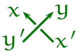
2) Yield knot and knotted surface invariants (*Carter et al. '01*):

(S,σ)-coloured diagram $(D,\mathcal{C})$ $\quad \& \quad \phi\colon S\times S \to \mathbb{Z}$ or $\mathbb{Z}_m$

$\rightsquigarrow$ Boltzmann weight $\mathscr{B}_\phi(\mathcal{C}) = \sum \phi(x,y) - \sum \phi(x,y).$





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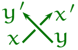

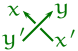

ϕ a 2-cocycle $\quad\Longrightarrow\quad$ a knot invariant given by

$$\{\, \mathscr{B}_\phi(\mathcal{C}) \mid \mathcal{C} \text{ is a } (S,\sigma)\text{-colouring of } D \,\}.$$

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A cohomology theory for YBE solutions should:

3) Unify cohomology theories for
   - associative structures,
   - Lie algebras,
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+ explain parallels between them (*L. '13*),

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4) Compute the Hochschild cohomology of $\Bbbk\,\mathrm{Mon}(S, \sigma)$ (an associative algebra associated to (S, σ); more on this tomorrow!).

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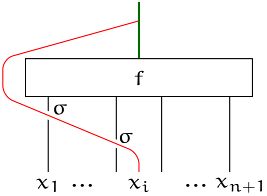

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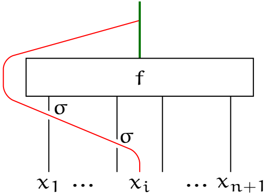

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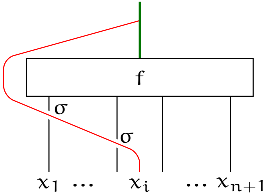

Theorem: ⤳ $d^{n+1} d^n = 0$;
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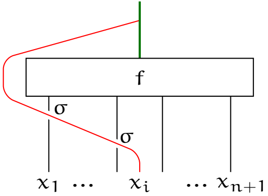

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➛ for “nice” M, a cup product $\smile\colon H^n \otimes H^m \to H^{n+m}$; …

3 A good theory?

1) & 2) For $\phi \in C^2(S, \sigma; \mathbb{Z}_{(n)})$,

$d^2\phi = 0 \implies \phi$ yields Boltzmann weights & diagonal deformations,

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braided cohomology of (S, σ) with coefs in M $\xleftarrow{\mathcal{QS}}$ Hochschild cohomology of $\Bbbk \operatorname{Mon}(S, \sigma)$ with coefs in M

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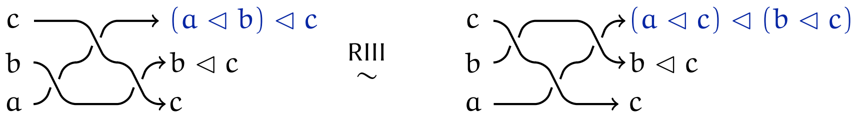
Open problem: How far is $\mathfrak{QS}$ from being an iso in general?

From now on, we concentrate on the cohomology of self-distributive structures. Much of what we will see works for any braided set, or even any linear YBE solution.

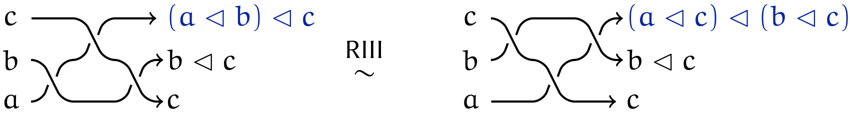
You Could Have Invented SD Cohomology If You Were...



diagrams:	D	$\overset{\text{R-move}}{\rightsquigarrow}$	D'
colourings:	$\mathcal{C}$	$\rightsquigarrow$	$\mathcal{C}'$
colouring sets:	$\mathrm{Col}_{S,\triangleleft}(D)$	$\overset{1:1}{\longleftrightarrow}$	$\mathrm{Col}_{S,\triangleleft}(D')$

Counting invariants: $\#\,\mathrm{Col}_{S,\triangleleft}(D) = \#\,\mathrm{Col}_{S,\triangleleft}(D')$.

4 … a Knot Theorist



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Question: Extract more information?

$$\omega(\mathcal{C}) = \omega(\mathcal{C}')$$

$$\Downarrow$$

$$\{\, \omega(\mathcal{C}) \,|\, \mathcal{C} \in \mathrm{Col}_{S,\triangleleft}(D) \,\} = \{\, \omega(\mathcal{C}') \,|\, \mathcal{C}' \in \mathrm{Col}_{S,\triangleleft}(D') \,\}.$$

Answer (*Carter–Jelsovsky–Kamada–Langford–Saito ’03*): State-sums over crossings, and Boltzmann weights:

$$\phi\colon S \times S \to \mathbb{Z}_m \qquad \rightsquigarrow \qquad \omega_\phi(\mathcal{C}) = \sum \pm\phi(a, b)$$

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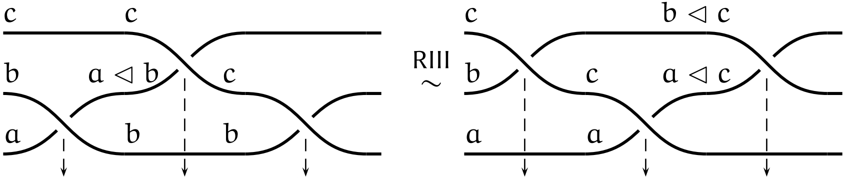
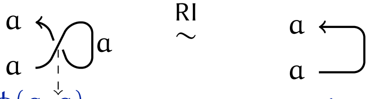


$$\phi(a,b)+\phi(a\triangleleft b,c)+\cancel{\phi(b,c)} = \cancel{\phi(b,c)}+\phi(a,c)+\phi(a\triangleleft c,b\triangleleft c)$$



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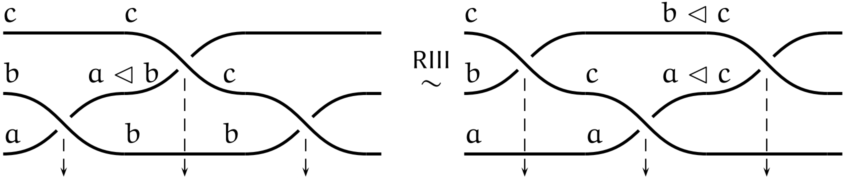

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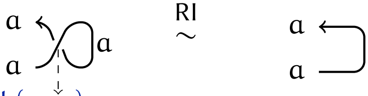

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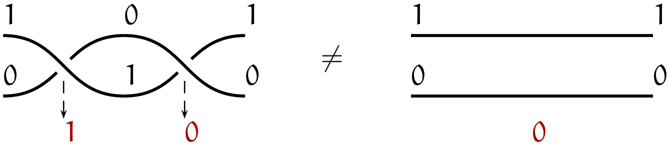
Quandle cocycle invariants $\supsetneq$ counting invariants.

Example: $S = \{0, 1\}$, $a \triangleleft b = a$,

$\phi(0,1) = 1$ and $\phi(a,b) = 0$ elsewhere.



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Generalisation: $K^n \hookrightarrow \mathbb{R}^{n+2}$ and $\phi\colon S^{\times(n+1)}\to\mathbb{Z}_m$.

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Wish:

$d^{n+1}\phi = 0 \implies \phi$ refines counting invariants for n-knots,

$\phi = d^n\psi \implies$ the refinement is trivial.

5 ... a Hopf Algebraist

Very open question: Classify nice Hopf algebras over $\mathbb{C}$.

Here "nice" = finite-dimensional pointed.

Applications:

- ✓ cohomology of H-spaces, e.g. Lie groups (*Hopf '41*);
- ✓ invariants of knots and 3-manifolds, TQFT;
- ✓ non-commutative geometry;
- ✓ condensed-matter physics, string theory,

....

5 ... a Hopf Algebraist

Very open question: Classify nice Hopf algebras over $\mathbb{C}$.
Here "nice" = finite-dimensional pointed.

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Examples:

- ✓ group algebras $\Bbbk G$;
- ✓ enveloping algebras of Lie algebras $U(\mathfrak{g})$;
- ✓ quantum groups: deformations $U_q(\mathfrak{g})$ for semisimple $\mathfrak{g}$,

....

Classification program (*Andruskiewitsch–Graña–Schneider '98*):

nice Hopf algebra A

$\rightsquigarrow$

Yetter–Drinfel'd module $V \in {}^{H}_{H}\mathbf{YD}$

✓ $G(A)$ = the group of group-like elements of A, $H(A) = \mathbb{C}G(A)$;

✓ $R(A)$ = coinvariants of $\mathrm{gr}(A) \twoheadrightarrow \mathrm{gr}(A)_0 = H(A)$, $V(A) = \mathrm{Prim}(R(A))$;

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YBE solution $(\mathbb{C}S, \sigma_{\lhd,\phi})$ $\leftsquigarrow$ rack $(S, \lhd)$ & $\phi\colon S \times S \to \mathbb{Z}_m$

$$\boxed{\sigma_{\lhd,\phi}\colon (a,b) \mapsto q^{\phi(a,b)}(b, a \lhd b)}$$

Here q is an mth root of unity, or transcendental.

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Wish:

$$d^2\phi = 0 \implies (\mathbb{C}S, \sigma_{\triangleleft,\phi}) \text{ is a braided vector space,}$$
$$\phi - \phi' = d^1\psi \implies \text{the braided vector spaces are isomorphic.}$$

... a Rack Theorist

Rack classification in 3 steps (*Joyce '82, Andruskiewitsch–Graña '03*):

1) Simple racks, i.e., without non-trivial quotients:

- ✓ permutation racks $S = \mathbb{Z}_p$, $a \lhd b = a + 1$, p prime;
- ✓ Alexander (= affine) racks $S = \mathbb{Z}_{p^k}$, $a \lhd b = ta + (1-t)b$,
  p prime, t generates $\mathbb{Z}_{p^k}$ over $\mathbb{Z}_p$;
- ✓ certain twisted conjugacy racks: subracks of $(G\,,\ a \lhd b = f(b^{-1}a)b)$,
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$\wr$ (various!) glueings

3) General racks.

A rack extension of S is a rack surjection $R \twoheadrightarrow S$.

If S is indecomposable, then $R \cong S \times_\alpha X$, which is $S \times X$ with

$$(a, x) \triangleleft (b, y) = (a \triangleleft b, \alpha(a, b, x, y)),$$

where X is a set, and $\alpha\colon S \times S \times X \times X \to X$ satisfies certain axioms.

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Important class: abelian rack extensions, i.e., with X an abelian group and

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Wish:

$$d^2\phi = 0 \implies \phi \text{ defines an abelian extension,}$$
$$\phi = d^1\psi \implies \text{the extension is trivial.}$$

7 The desired cohomology theory

Fenn et al. '95 & Carter et al. '03 & Graña '00:

Shelf $(S, \triangleleft)$ & abelian group X $\rightsquigarrow$ cochain complex

$$C_R^k(S, X) = \mathrm{Map}(S^{\times k}, X),$$

$$(d_R^k f)(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} (f(a_1, \ldots, \widehat{a_i}, \ldots, a_{k+1}) - f(a_1 \triangleleft a_i, \ldots, a_{i-1} \triangleleft a_i, a_{i+1}, \ldots, a_{k+1}))$$

$\rightsquigarrow$ Rack cohomology $H_R^k(S, X) = \mathrm{Ker}\, d_R^k / \mathrm{Im}\, d_R^{k-1}$.

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Quandle $(S, \lhd)$ & abelian group X $\rightsquigarrow$ sub-complex of (C_R^k, d_R^k):

$$C_Q^k(S, X) = \{ f \colon S^{\times k} \to X \mid f(\ldots, a, a, \ldots) = 0 \}$$

$\rightsquigarrow$ Quandle cohomology $H_Q^k(S, X)$.

$$C^k_R(S,X) = \mathrm{Map}(S^{\times k}, X),$$

$$(d^k_R f)(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} \big(f(a_1, \ldots, \widehat{a_i}, \ldots, a_{k+1}) - f(a_1 \lhd a_i, \ldots, a_{i-1} \lhd a_i, a_{i+1}, \ldots, a_{k+1})\big);$$

$$C^k_Q(S,X) = \{ f \colon S^{\times k} \to X \mid f(\ldots, a, a, \ldots) = 0 \}.$$

In small degree:

$(d^0_R f)(a_1) = 0$

$(d^1_R f)(a_1, a_2) = f(a_1 \lhd a_2) - f(a_1)$

$H^1_R(S,X) \cong \mathrm{Map}(\mathrm{Orb}(S), X)$

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$(d^2_R f)(\overline{a}) = f(a_1 \triangleleft a_2, a_3) - f(a_1, a_3) + f(a_1, a_2) - f(a_1 \triangleleft a_3, a_2 \triangleleft a_3)$

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Remark: $d^2_R d^1_R = 0 \iff$ self-distributivity for $\lhd$.

This is what we were looking for! This construction yields:

- ✓ Boltzmann weights for constructing higher knot invariants;
- ✓ an important class of braided vector spaces giving nice Hopf algebras;
- ✓ a parametrisation of abelian rack extensions.

8 Topological realisation

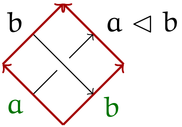
Fenn–Rourke–Sanderson '95:

Shelf $(S, \triangleleft)$ $\rightsquigarrow$ rack (= classifying) space $B(S)$. It is a CW-complex:

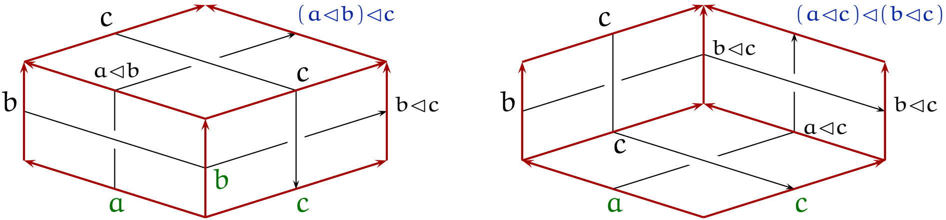
deg 0: $*$

deg 1: $* \xrightarrow{a} *$

deg 2:



deg 3:



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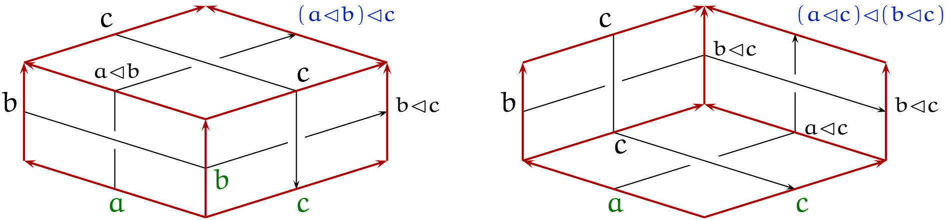
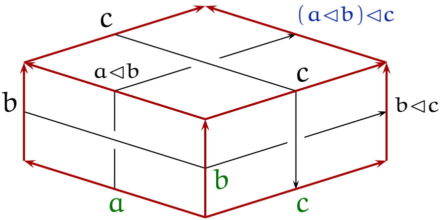
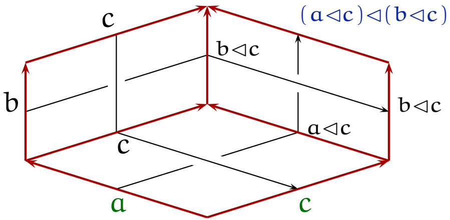


Remark: the edges can be colored starting from the green corner
$\iff$ $\triangleleft$ is self-distributive.

deg 3:





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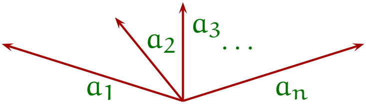
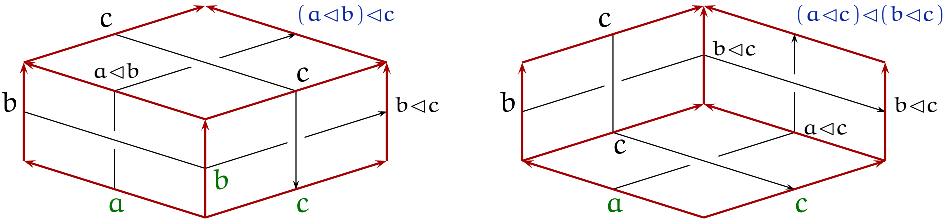
deg n: $\coprod_{S^{\times n}} [0,1]^n$



The colouring continues uniquely to other edges of $[0,1]^n$.

deg 3:



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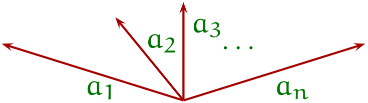

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Boundaries: usual topological ones.

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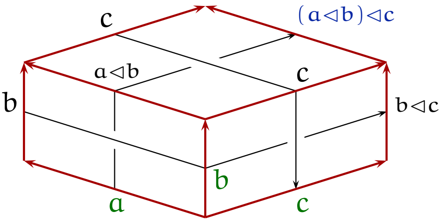

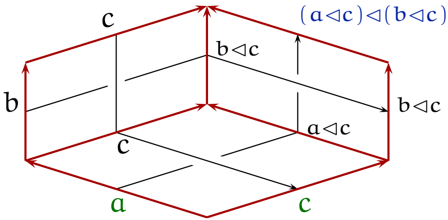

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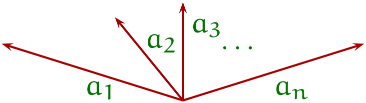

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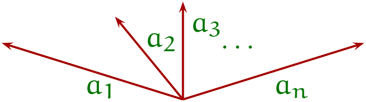

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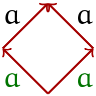
Nosaka '11: To get quandle cohomology, add 3-dimensional cells bounding



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So, rack spaces bring topological tools in the study of H^*_R.

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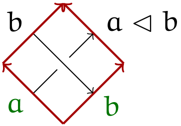
✓ $\boxed{\pi_1(B(S)) \cong As(S)}$ where $As(S) := \langle S \mid a\,b \;=\; b\,(a \lhd b)\rangle$ is the associated (= adjoint = structure = universal enveloping) group of $(S, \lhd)$.



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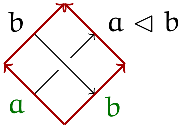

✓ Rack cohomology becomes a pre-cubical cohomology, i.e.,

$$d^k_R = \sum_{i=1}^{k+1} (-1)^{i-1} (d^k_{i,0} - d^k_{i,1}), \qquad d_{i,\varepsilon} d_{j,\zeta} = d_{j-1,\zeta} d_{i,\varepsilon} \quad \text{for all } i < j.$$

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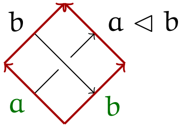

✓ Rack cohomology becomes a pre-cubical cohomology, i.e.,

$$d^k_R = \sum_{i=1}^{k+1} (-1)^{i-1}(d^k_{i,0} - d^k_{i,1}), \qquad d_{i,\varepsilon} d_{j,\zeta} = d_{j-1,\zeta} d_{i,\varepsilon} \quad \text{for all } i < j.$$

✓ Concrete computations (*Fenn–Rourke–Sanderson '07*):

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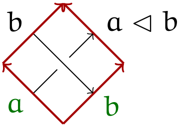

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2) Free rack on n generators FR_n: $B(FR_n) \cong \vee_n \mathbb{S}^1$.

9 Graphical interpretation

$$C_R^k(S, \mathbb{Z}_m) = \mathrm{Map}(S^{\times k}, \mathbb{Z}_m),$$

$$(d_R^k f)(a_1, \ldots, a_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i-1} \big(f(a_1, \ldots, \widehat{a_i}, \ldots, a_{k+1}) - f(a_1 \triangleleft a_i, \ldots, a_{i-1} \triangleleft a_i, a_{i+1}, \ldots, a_{k+1}) \big)$$

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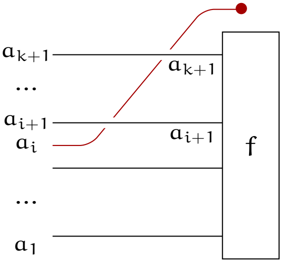

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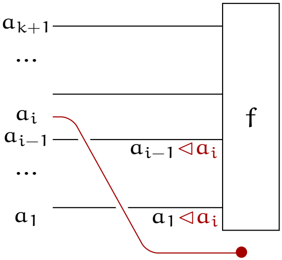
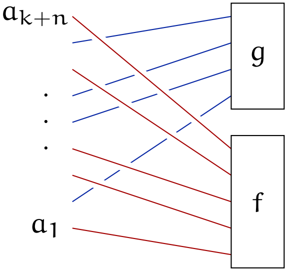


$\Bigg)$

10 Cup product

$$\smile : C^k_R \otimes C^n_R \to C^{k+n}_R$$

$$f \smile g(a_1, \ldots, a_{k+n}) = \sum_{\text{splittings}} (-1)^{\#\times}$$



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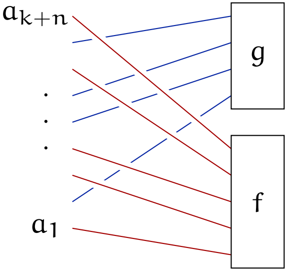

Theorem:

- ✓ $(C^*_R, \smile)$ d.g. associative, graded commutative up to an explicit htpy;
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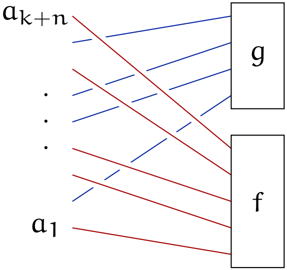

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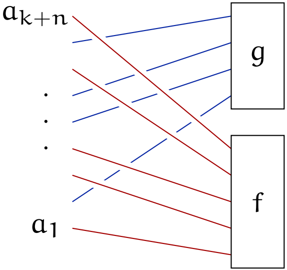

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Interpretations:

- ✓ quantum shuffle coproduct;
- ✓ topological cup product;
- ✓ cup product in cubical cohomology;
- ✓ shelf $\rightsquigarrow$ explicit d.g. bialgebra $\rightsquigarrow$ cohomology.

(*Serre '51*, *Baues '98*, *Clauwens '11*, *Covez '12*, *L. '17*, *Covez–Farinati–L.–Manchon '19*.)

11 Betti numbers

$$C^k_R(S,X) = \mathrm{Map}(S^{\times k}, X),$$

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Theorem (*Etingof–Graña '03*): If $(S,\triangleleft)$ is a rack and , then

$$\boxed{H^k_R(S,X) \cong \mathrm{Map}(\mathrm{Orb}(S)^{\times k}, X)} \qquad \text{i.e., } b_k(S) = |\,\mathrm{Orb}(S)|^k$$

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Solution: Take $X = \mathbb{Z}_p$, or the p-torsion of $H^k_R(S,\mathbb{Z})$, where $p \mid \#\,\mathrm{Inn}(S)$.

12 Quandle cohomology vs rack cohomology

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Theorem (*Litherland–Nelson '03*): The rack cohomology of a quandle splits:

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13 Quandle cohomology is a Quillen cohomology

Szymik '19

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$$\mathrm{As}(S) := \langle S \mid a\, b \;=\; b\,(a \triangleleft b) \rangle$$

Theorem (*Joyce '82*): One has a pair of adjoint functors

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Theorem. There is a graded algebra morphism $HH^*(\mathrm{As}(S), X) \to H^*_R(S, X)$.

Interpretations:

✓ explicit map: quantum symmetriser (*Covez '12*, *Farinati & García Galofre '16*);

✓ $B(S) \to B(\mathrm{As}(S))$ (*Fenn–Rourke–Sanderson '95*).