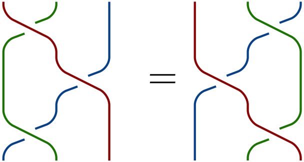
Yang–Baxter Equation III

Victoria LEBED, Université Caen Normandie



$$(ab)c = a(bc)$$

$$(a \lhd b) \lhd c =$$
$$(a \lhd c) \lhd (b \lhd c)$$

T-Days III, Caen, October 2019

1 Associative invariants

The structure group of a braided set (S, σ):

$$\boxed{\mathrm{Grp}(S, \sigma) = \langle\ S \mid xy = y'x' \text{ whenever } \sigma(x, y) = (y', x')\ \rangle}$$

Structure monoids are defined similarly.

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✓ Lie algebra $(V, [\,], 1)$, $\sigma(x \otimes y) = y \otimes x + \hbar 1 \otimes [x,y]$,

$$\mathrm{UEA}(V, [\,]) \simeq {}^{\Bbbk\,\mathrm{Mon}(S,\sigma)}/_{1 = 1_{\mathrm{Mon}}};$$

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✓ shelf $(S, \triangleleft)$, $\sigma(x, y) = (y, x \triangleleft y)$,

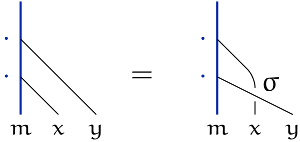
$$\mathrm{Grp}(S, \sigma) = \mathrm{Grp}(S, \triangleleft).$$

2 Representations

$$\mathrm{Grp}(S,\sigma) = \langle\, S \mid xy = y'x' \text{ whenever } \sigma(x,y) = (y',x') \,\rangle$$

Representations of (S,σ) := representations of $\Bbbk\,\mathrm{Mon}(S,\sigma)$,
i.e. vector spaces M with $M \times S \xrightarrow{\cdot} M$ s.t.

$$(m \cdot x) \cdot y \quad = \quad (m \cdot y') \cdot x'$$



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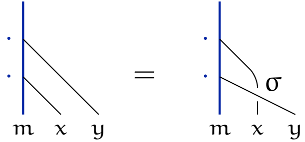

Examples:

- trivial rep.: $M = \Bbbk$, $m \cdot x = m$;
- $M = \Bbbk\,\mathrm{Mon}(S,\sigma)$, $m \cdot x = mx$;
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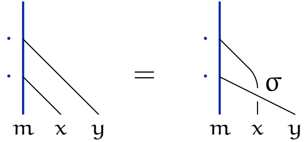

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Remark: The coefficients for yesterday's cohomology theory are (S,σ)-bimodules in this sense.

3 Associative invariants: Involutive case

The structure group of a braided set (S,σ):

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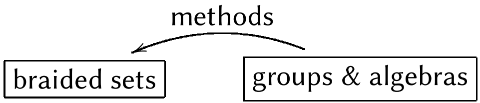

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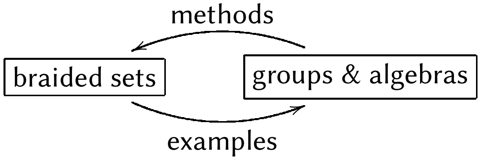

Theorem: (S, σ) a "nice" finite braided set, $\sigma^2 = \mathrm{Id} \implies$

✓ $\mathrm{Mon}(S, \sigma)$ is of I-type, cancellative, Ore;

✓ $\mathrm{Grp}(S, \sigma)$ is solvable, Garside, Bieberbach;

✓ $\Bbbk\,\mathrm{Mon}(S, \sigma)$ is Koszul, noetherian, Cohen–Macaulay, Artin–Schelter regular

(*Manin, Gateva-Ivanova & Van den Bergh, Etingof– Schedler–Soloviev, Jespers–Okniński, Chouraqui, Bachiller–Cedó –Vendramin,…80'-…*).

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✓ $\Bbbk\,\mathrm{Mon}(S,\sigma)$ is Koszul, noetherian, Cohen–Macaulay, Artin–Schelter regular;

✓ $\mathrm{Grp}(S,\sigma)$ is bi-orderable $\Leftrightarrow$ free abelian $\Leftrightarrow$ (S,σ) is trivial;

✓ $\mathrm{Grp}(S,\sigma)$ is left-orderable $\Leftrightarrow$ poly-$\mathbb{Z}$ $\Leftrightarrow$ (S,σ) is MP.

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Example: $S = \{a, b\}$, $aa \overset{\sigma}{\longleftrightarrow} bb$, $ab \circlearrowleft \sigma$, $ba \circlearrowleft \sigma$;

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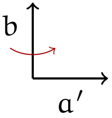
Realisation by Euclidean transformations of $\mathbb{R}^2$:



$$\begin{array}{c} b = a'ba' \\ \big\downarrow \; {\scriptstyle a = a'b} \\ a^2 = b^2 \end{array}$$

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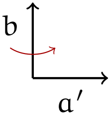

$$b = a'ba'$$
$$\downarrow a = a'b$$
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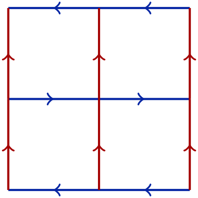
$\mathbb{R}^2/G \cong$ Klein bottle:



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$$\boxed{\mathrm{Grp}(S, \lhd) = \langle\ S \mid xy = y(x \lhd y)\ \rangle}$$

<u>Theorem</u> (*L.–Vendramin '19*): $(S, \lhd)$ a finite rack $\Longrightarrow$ dichotomy for $G := \mathrm{Grp}(S, \lhd)$:

① $G \simeq \mathbb{Z}^r, \quad r = \#\,\mathrm{Orb}(S, \lhd)$;

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Questions:

① Characterise such racks?

② Better understand such groups?

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✓ G abelian $\Longrightarrow$ no torsion in $H^2(S, \lhd, M)$
(abelian quandles may have torsion).

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A tool: For any finite rack, a map

$$\prod_{i=1}^{r} \mathrm{Stab}(x_i, \mathrm{Grp}(S, \lhd)) \twoheadrightarrow H_2(S, \lhd).$$

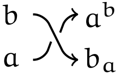
Example: $(a, bc^{-1}d) \mapsto$

$$a \xrightarrow{\ b\ } a \lhd b \xleftarrow{\ c\ } (a \lhd b) \mathbin{\widetilde{\lhd}} c \xrightarrow{\ d\ } ((a \lhd b) \mathbin{\widetilde{\lhd}} c) \lhd d\ .$$

5 SD invariants

Birack = braided set (S, σ) with $\sigma(a, b) = (b_a, a^b)$ s.t.

- ✓ σ is invertible;
- ✓ the maps $\forall b,\ a \mapsto a^b$ and $a \mapsto a_b$ are invertible.



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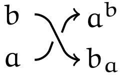

<u>Theorem</u> (*Soloviev & Lu–Yan–Zhu '00, L.–Vendramin '17*):

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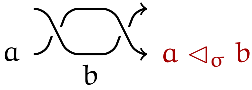
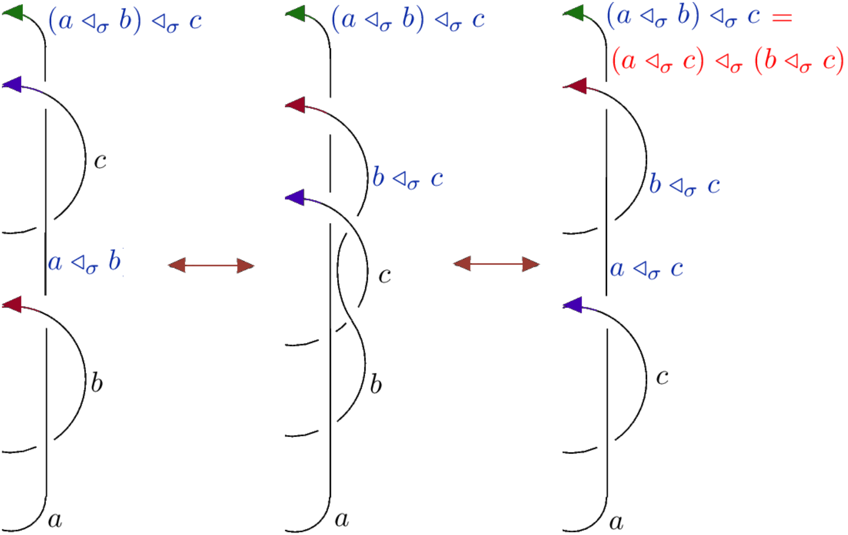


Proof of the self-distributivity of $\lhd_\sigma$:



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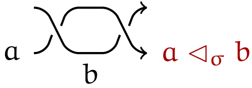

✓ This is a projection **Birack** ↠ **Rack** along involutive biracks:

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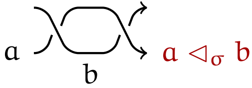

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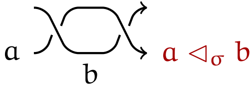

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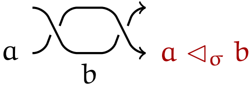

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$(S,\sigma) \not\cong (S,\sigma')$ as biracks!!!